

The Shape of a Ponytail and the Statistical Physics of Hair Fiber Bundles

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A general continuum theory for the distribution of hairs in a bundle is developed, treating individual fibers as elastic filaments with random intrinsic curvatures. Applying this formalism to the iconic problem of the ponytail, the combined effects of bending elasticity, gravity, and orientational disorder are recast as a differential equation for the envelope of the bundle, in which the compressibility enters through an ‘equation of state’. From this, we identify the balance of forces in various regions of the ponytail, extract a remarkably simple equation of state from laboratory measurements of human ponytails, and relate the pressure to the measured random curvatures of individual hairs.

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One of the most familiar features of a bundle of hair such as a ponytail is its ‘body’ or ‘volume.’ Close examination reveals that this property arises in a subtle way from the stiffness and shapes of the individual fibers, whose meandering paths through the bundle produce many collisions with other hairs (Fig. 1). These meanderings are in part a consequence of the contacts themselves, but hairs also have an *intrinsic* waviness or curliness [1, 2]. Such curvatures may be generated during growth, and vary with ethnicity. They are clearly also modified by chemical, thermal, and mechanical forces, as in the ‘water wave’ treatment, or a ‘perm’ [1].

From Leonardo to the Brothers Grimm our fascination with hair has endured in art and science [3, 4]. Yet, we still do not have an answer to perhaps the simplest question that captures the competing effects of filament elasticity, gravity, and mutual interactions: *What is the shape of a ponytail?* Note that the average human has $\sim 10^5$ head hairs, so if even a modest fraction is gathered into a ponytail, the number involved is enormous: this is a problem in statistical physics.

Here we propose a theory for the ponytail shape on the basis of a continuum theory for the spatial distribution of hairs in a bundle. Their random curvatures give rise to a swelling pressure characterized by an ‘equation of state’ (EOS) of hair, a concept first introduced semi-empirically by van Wyk in 1946 in relation to the compressibility of wool [5, 6], and explored recently for two-dimensional randomly-curved fibers by Beckrich *et al.* [7]. We exploit the predominantly vertical alignment of hairs and axisymmetry to justify a number of approximations that render the problem analytically tractable, and thereby reduce the many-body problem to a one-body problem for the ponytail *envelope*. This shows how the EOS modifies the envelope shape from that of a single hair bent by gravity, a classic problem in elasticity [2].

In parallel with the theoretical development, we measured the shapes of ponytails made from commercially

available hair ‘switches’ [8], and of their component fibers. Typical human hair has an elliptical cross section and a distribution of major axis diameters $40 \lesssim d \lesssim 140 \mu\text{m}$. We found $d = 79 \pm 16 \mu\text{m}$ for a random sample from the switches. Hair has an average density of $\simeq 1.3 \text{ g/cm}^3$ [1], and a linear mass density $\lambda \simeq 65 \mu\text{g/cm}$ (in more amusing units, 6.5 g/km). Though its internal microstructure is complex, the bend and twist moduli of hairs [1] are consistent with those of a homogeneous incompressible material with a nylon-like modulus $E \approx 4 \text{ GPa}$. On the centimeter scale classical filament elasticity holds, with a bending modulus $A = E\pi d^4/64 \approx 8 \times 10^{-9} \text{ N m}^2$. The quantities λ and A and the acceleration of gravity g combine to form the length $\ell = (A/\lambda g)^{1/3} \approx 5 \text{ cm}$ on which gravity bends a hair [2].

Individual hairs display a range of shapes (Fig. 2a) which we have quantified by high-resolution stereoscopic imaging [9]. Both the mean squared curvature

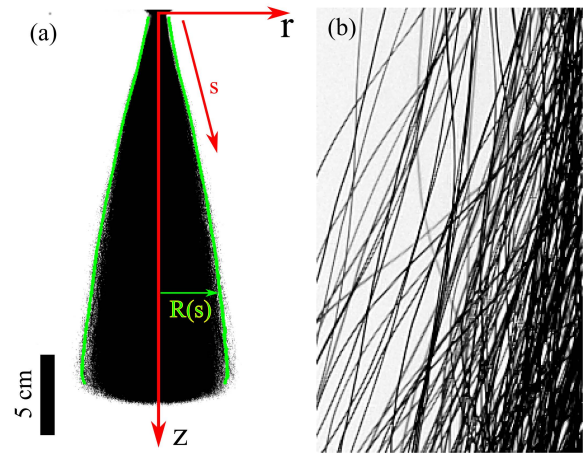


FIG. 1: (color online) A ponytail. (a) Rotationally-averaged image of a switch of $N \approx 9500$ fibers, approximately 25 cm long. Coordinate system for envelope shape $R(s)$ in terms of arc length $s(z)$. (b) Meanderings of hairs near ponytail edge.

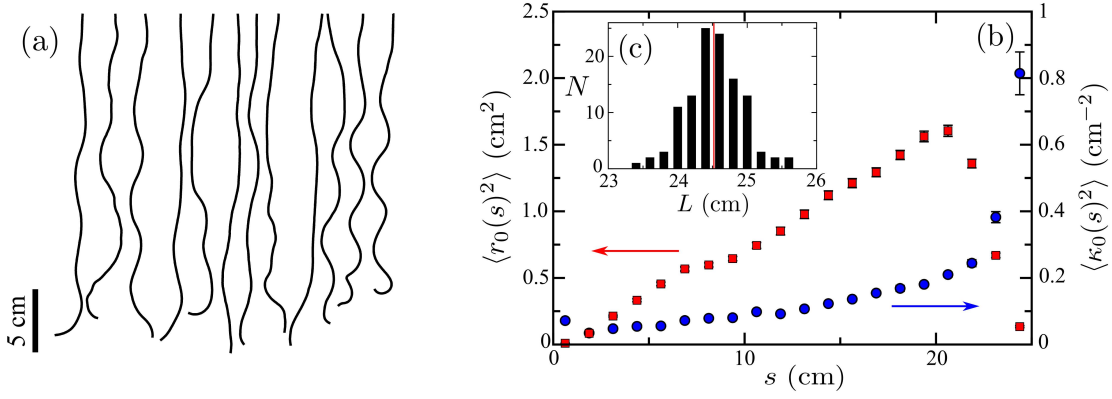


FIG. 2: (color online) Geometry of hairs. (a) Representative projections of hair contours, thickened for clarity. (b) Mean squared radial excursion and curvature as functions of arc length, from processing stereoscopic image pairs [9]. The reconstructed arc lengths cluster tightly around 24.50 ± 0.05 cm (inset histogram), demonstrating the accuracy of the image processing and analysis methodology. Error bars are standard errors from ensemble averaging ($N = 115$ fibers in total).

$\langle \kappa_0(s)^2 \rangle$ and the radial excursion $\langle r_0(s)^2 \rangle$ increase with arclength s measured from the top of the switch (Fig. 2b). Whilst some of this is undoubtedly due to gravity (recall $\ell \approx 5$ cm), the major part is intrinsic, as we have verified by examining inverted hairs. This is in part due to the preparation process: after washing and rinsing, the hairs in a drying ponytail pass through a glass transition with decreasing humidity [1], locking in the intrinsic curvature [10], which is naturally reduced in the vicinity of the clamp due to confinement by neighboring filaments. Although this is something of a complication when it comes to interpreting the results, we must regard it as an essential feature of hair switches and ponytails comprised of *real* fibers. For later reference, the length-wise averages are $\langle \kappa_0^2 \rangle = 0.15 \pm 0.01 \text{ cm}^{-2}$ and $\langle r_0^2 \rangle = 0.80 \pm 0.05 \text{ cm}^2$.

Figure 3a shows measured profiles of radius $R(z)$ vs distance z below the clamp for four separate switches of length $L \approx 25$ cm. Each profile has been obtained from the rotational average of five images as in Fig. 1a viewed from angles 72° apart. The switch profile shows quite good reproducibility and is well modeled and explained by the theory we now describe. Our starting point for the continuum theory is to introduce the fiber length density $\rho(\mathbf{r})$ (the number of fibers per unit area intersecting a plane perpendicular to the fibers) and the mean fiber tangent vector $\mathbf{t}(\mathbf{r})$, the local average of unit vectors along the fibers. The latter is a meaningful quantity when the fiber orientation remains coherent over length scales much larger than the mean fiber spacing $\rho^{-1/2}$. Here the fibers are indeed well-aligned, with $t \equiv |\mathbf{t}| \approx 1$, unlike in non-woven fabrics [11]. In the absence of fiber ends in the bulk these continuum fields obey a continuity equation $\nabla \cdot (\rho \mathbf{t}) = 0$. The analogy to the continuity equation of fluid mechanics mathematizes the remark made by Leonardo at the beginning of the 16th century, that hair resembles fluid streamlines [3], an observation

which has been exploited in more recent times to aid computer animation [4]. For later use we also define the local packing fraction $\phi = \pi \rho d^2/4$. We propose the energy of an axisymmetric fiber bundle is

$$\mathcal{E}[\rho, \mathbf{t}] = \int d^3 \mathbf{r} \rho \left(\frac{1}{2} A \kappa^2 + \varphi(\mathbf{r}) + \langle u \rangle \right), \quad (1)$$

where $\kappa = |(\mathbf{t} \cdot \nabla) \mathbf{t}|$ is the curvature field. The terms in (1) are the elastic energy of mean curvature, the external (*e.g.* gravitational) potential φ , and a fiber confinement energy per unit length $\langle u \rangle$ that aggregates all terms involving disorder, such as contacts and natural curvatures. Without axisymmetry, (1) should include terms arising from the torsion of \mathbf{t} . As in density functional theory [12], we suppose that $\langle u \rangle$ is some *local* function of ρ . Minimization of (1) provides a variational principle for the bundle shape and the distribution of fibers. When recast as mechanical force balance we make contact with the EOS, and identify $P(\rho) = \rho^2 d\langle u \rangle / d\rho$ as the pressure.

To address the specific problem of ponytail shapes we now introduce models which allow for largely analytical calculations. With axisymmetry, an integrated form of the continuity equation is $2\pi r \rho \sin \theta = -\partial n / \partial z$ and $2\pi r \rho \cos \theta = \partial n / \partial r$, where $n(r, z)$ is the number of fibers within a radius r at depth z and θ is the angle the tangent vector makes to the vertical. ($n(r, z)$ plays the role of the stream function in fluid mechanics.) We insert this into Eq. (1), and use a trial *uniform* radial density function with *self-similar* form, $n(r, z) = N[r/R(z)]^2$ where N is the total number of fibers and $R(z)$ is the ponytail radius (Fig. 1a). In practice it is more convenient to use $R(s)$, where $s(z)$ is the arclength from the clamp. If L is the fully-extended hair length and $\varphi = \lambda g z$ the gravitational potential energy, then to second order in $R_s \equiv dR/ds$, neglecting a small splay term, one finds

$$\mathcal{E} = N \int_0^L ds \left[\frac{1}{2} \tilde{A} R_{ss}^2 + \frac{1}{2} \tilde{\lambda} g (L - s) R_s^2 + \langle u \rangle \right], \quad (2)$$

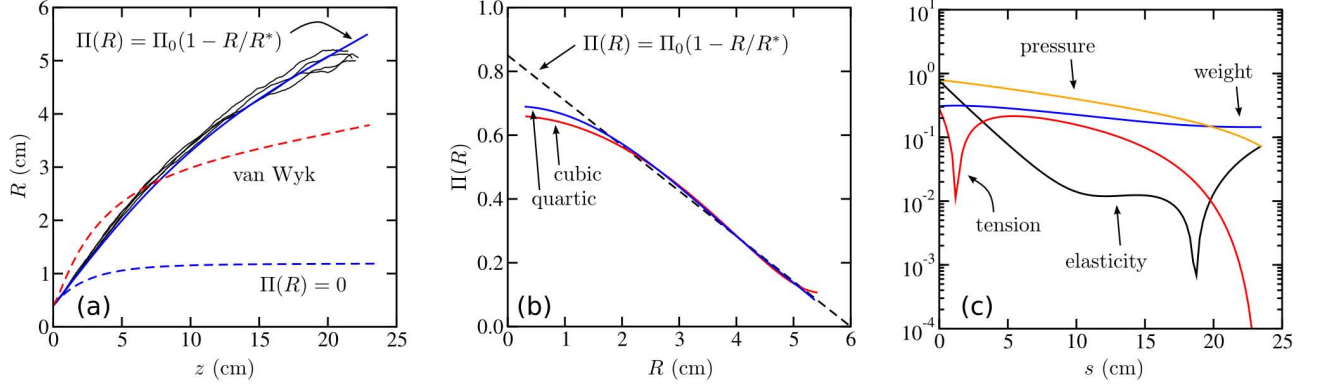


FIG. 3: (color online) Analysis of ponytail shapes. (a) Measured hair switch profiles (thin black lines), compared to the prediction of Eq. (3) with $\Pi(R)$ as given (solid blue line), with $\Pi(R) = 0$ (dashed blue line), and with the van Wyk EOS (dashed red line) [5, 13] (b) Dimensionless swelling pressure $\Pi(R)$ from cubic and quartic fits to the measured profiles, using procedure outlined in the text. (c) The magnitude of the four terms in Eq. (3) for the calculated profile (solid blue line) in (a).

where $\langle u \rangle$ depends on $\bar{\rho} = N/(\pi R^2)$. The problem is now mapped to an equivalent single fiber hanging under gravity in the presence of a radial force field derived from $\langle u \rangle$. The uniform distribution in the trial density function yields renormalized material properties $\tilde{A} = A/2$ and $\tilde{\lambda} = \lambda/2$. Minimizing Eq. (2) leads to

$$\ell^3 R_{ssss} - (L - s)R_{ss} + R_s - \Pi(R) = 0 \quad (3)$$

where $\Pi(R) = 4\ell^3 P/A\bar{\rho}R = -(2\ell^3/A) d\langle u \rangle/dR$. We term this the *ponytail shape equation*. It describes a force balance on a length element of the notionally equivalent single fiber as the sum of four dimensionless terms which are, respectively, an elastic restoring force, a ‘string tension’ contribution, a weight term, and a radial swelling force corresponding to a pressure gradient P/R per unit fiber density. The ratio $Ra \equiv L/\ell$ we shall term the *Rapunzel number*, since it is a dimensionless measure of the ponytail length. When the ponytail hangs from a circular clamp of radius R_c , the boundary conditions are $R(0) = R_c$ and $R_s(0) = \tan \theta_c$ where θ_c is the ‘launch’ angle of the outermost fibers emerging from the clamp. At the free bottom of the ponytail the boundary conditions are $R_{ss}(L) = R_{sss}(L) = 0$. To the order at which we are working, (3) is supplemented by $z_s \simeq 1 - R_s^2/2$ to give the parametric ponytail shape $(z(s), R(s))$.

Fitting the above theory to the experimental ponytail profiles in Figure 3a reveals a remarkably simple form for the pressure $\Pi(R)$. While the full Eq. (3) can in principle be used to determine $\Pi(R)$ from the profiles, the extraction of high-order derivatives from such data is notoriously problematic. We notice though that, away from the clamp, R_{ssss} is likely to be subdominant to the other terms in Eq. (3) and therefore we can neglect this elastic term and approximate $\Pi(R) \simeq R_s - (L - s)R_{ss}$, where the right-hand-side is obtained by a low-order polynomial fit to the data. Figure 3b shows that

in this region the EOS is accurately represented by

$$\Pi(R) = \Pi_0(1 - R/R^*), \quad (4)$$

with $\Pi_0 \approx 0.85$ and $R^* \approx 6$ cm. Inserting this into Eq. (3) and now *including* the elastic term recovers the solid blue line in Fig. 3a, in excellent agreement with the data (by contrast the van Wyk EOS simply cannot be made to fit the data [13]). In making these calculations we use $R_c \approx 4$ mm and $\theta_c \approx 17^\circ$, obtained from measurements near the clamp. The starting radius R_c corresponds to $\phi \approx 0.95$, consistent with the near close packing of the fibers, whilst the starting angle θ_c is presumably governed by the method of clamping (in our case a rubber band wrapped several times around the top of the switch).

Figure 3c shows the magnitudes of the terms in Eq. (3). In the region near the clamp ($s \lesssim 2$ cm), elasticity and pressure balance, but for the most part the dominant balance is between weight and pressure, justifying our claim that the elastic term is subdominant away from the clamp. The blue dashed line in Fig. 3a is the profile for $\Pi(R) = 0$. Since $Ra \approx 5$ is quite large, this shape is dominated by gravity. Comparing the dashed and solid blue lines in Fig. 3a highlights again the dominant role played by the swelling pressure in determining the shape.

Given $\Pi(R)$, the shape of any ponytail can be predicted. Thus we are led to a kind of *experimentum crucis*, shown in Fig. 4, in which the predictions of Eq. (3) are compared to the profiles of progressively cut hair switches. The agreement is very good. We observe empirically that the launch angle θ_c is remarkably constant at 17° , only decreasing to 16° for the shortest hair switch (all calculations used $\theta_c = 17^\circ$). The calculated profiles show a modest compaction on increasing length, while the experimental profiles almost completely collapse on top of one another. This is not an effect of plasticity [11] since the switches are compressed in the cutting stage. The predicted profiles can similarly collapse (Fig. 4c), by

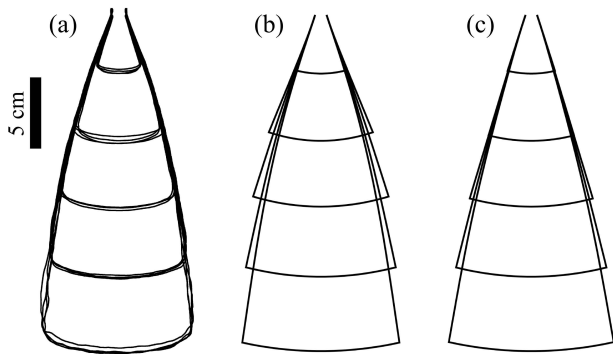


FIG. 4: Trimming a ponytail. (a) Superimposed rotationally-averaged outlines of four hair switches, cut down from 25 cm in steps of 5 cm. (b) Predicted profiles from Eq. (3) with $\Pi(R) = \Pi_0(1 - R/R^*)$. (c) Predicted profiles with $\Pi(R, s) = \Pi_0(1 - R/R^*) \times 2s/L^*$ where $L^* = 25$ cm.

allowing for an additional length dependence reflecting the gradient in the intrinsic fiber properties (Fig. 2b).

How are we to interpret the EOS recovered by this analysis? We propose that it can be associated with the intrinsic curvatures of the filaments. Let us imagine that the effects of collisions with neighboring fibers can be captured by a *tube model*. Specifically, consider a helical fiber [14] of radius a_0 confined within a cylinder of radius $a < a_0$, for which $\langle u \rangle \approx A\langle \kappa_0^2 \rangle (1 - a/a_0)^2/2$ (this is not exact but is quite accurate). Matching this to $\langle u \rangle = (A/2\ell^3) \int_R^\infty \Pi(R) dR$, obtained by integrating the expression below Eq. (3), and inserting our empirical $\Pi(R)$ gives $a/a_0 \approx 1 - \alpha + \alpha R/R^*$ where $\alpha = \sqrt{\Pi_0 R^*/2\ell^3 \langle \kappa_0^2 \rangle} \approx 0.4$. If we additionally suppose $a_0^2 \approx \langle r_0^2 \rangle$ then for instance at the clamp ($R = R_c$) the confining tube radius $a \approx 6$ mm. This seems to be in reasonable accord with observations, (see *e.g.* Fig. 1b). In the tube model, the pressure thus arises from increasing confinement of the fibers but they are still far from being completely straightened, even at high compression. This is perhaps not surprising considering the role that must eventually be played by friction.

Of the existing EOS theories, that of van Wyk does not fit our data (red dashed line, Fig. 3a), nor does it have an explicit link to the random curvatures. The fiber-collision model [7] can be extended to three dimensions, but the link to the underlying statistical properties becomes very unwieldy. More importantly, in that model fiber excursions are limited by *nearest-neighbor* collisions. This is not necessarily the case in three dimensions, and in fact is not supported by our data. Hence the microscopic link between fiber confinement and packing fraction remains an important open problem. Interestingly both the tube model and the collision model predict that the pressure remains finite on approach to the close packing limit, in marked contrast to thermal systems of hard particles. Thus, a bundle can be collapsed by sufficiently strong

inter-fiber attractions, such as the capillary forces acting on wet hair [15] or a paintbrush.

The program laid out here extends some central paradigms in statistical physics to the enchanting problem of ponytail shapes. The remarkably simple equation of state we have found, along with the systematic variation of intrinsic curvature along fibers, may open the way to understanding a wide range of hair and fur geometries. It is also of interest to extend the analysis to the *dynamics* of fiber bundles, epitomized by the ‘swing’ of a ponytail [16], where the notion of an equivalent single fiber may again prove fruitful.

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